Solving Monotone Variational Inequalities with Best Response Dynamics

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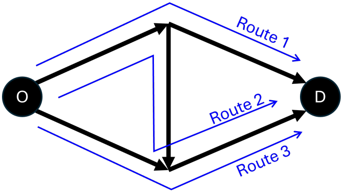
slides / paper

Population Games

- a large number of anonymous players
- a strategy set $\mathcal{S} = \{1, \cdots, n\}$
- a social state $x = (x_1, \cdots, x_n) \in \Delta := \{x_i \geq 0, \sum_i x_i = 1\}$
- a cost function $F : \Delta \to \mathbb{R}^n$ maps a social state to costs
- an incentive-oriented learning rule [1] $\dot{x} = g(x)$
- Nash Equilibrium:

$$\text{NE}(F) = \{x^* \in \Delta : x_i^* > 0 \implies F_i(x^*) \leq F_j(x^*), \quad \forall i, j \in \mathcal{S}\}$$

$\Rightarrow$ incentive-oriented evolution process to seek NE



[1] Best Response, Pairwise Comparison, Imitation, Excess Payoff

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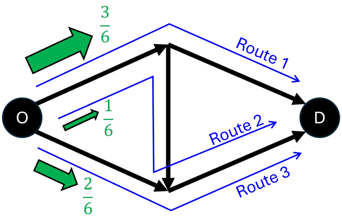

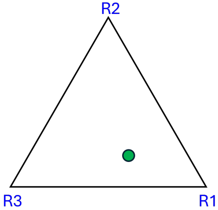

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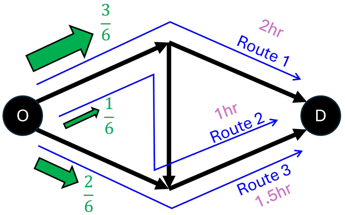

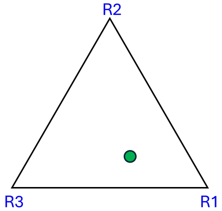

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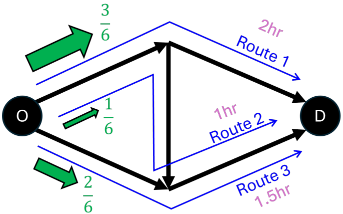

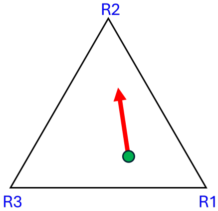

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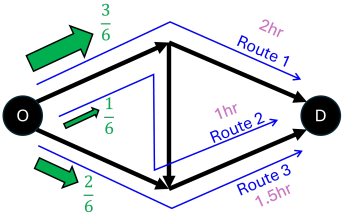

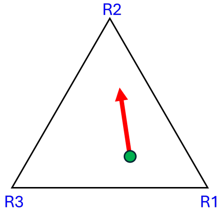

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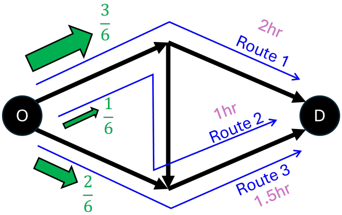

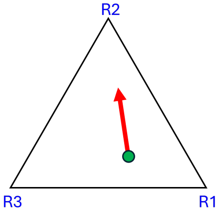

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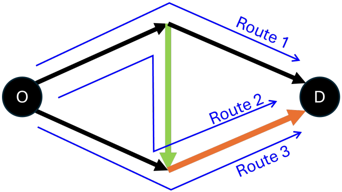
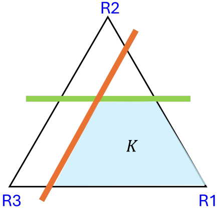
Overview

- Problem:
  Generalize population games from standard Δ to general set K to support scenarios having additional constraints on the social state.





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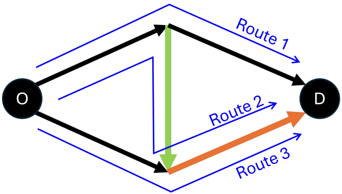
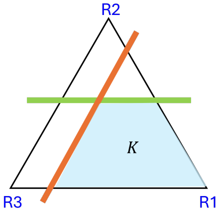
  - finding NE within set K is a Variational Inequality (VI) problem
  - most algorithms for solving VIs do not capture user behaviors!





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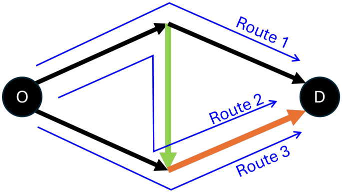
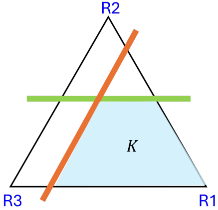
- Contribution:
  - We leverage best response dynamics to solve variational inequalities.
  - We analyze the robustness of this incentive-oriented algorithm.





Finding NE is a Variational Inequality problem

Definition (Variational Inequality)

Given a set $K \subseteq \mathbb{R}^n$ and a mapping $F : K \to \mathbb{R}^n$, we say that $x^* \in K$ solves the variational inequality, denoted as $\mathsf{VI}(K, F)$, if

$$(y - x^*)^T F(x^*) \geq 0, \quad \forall y \in K.$$

The set of all solutions is denoted as $\mathsf{SOL}(K, F)$.

- In standard population game F:
$$\mathsf{NE}(F) = \{x^* \in \Delta : x_i^* > 0 \implies F_i(x^*) \leq F_j(x^*), \quad \forall i, j \in \mathcal{S}\}$$
$$= \{x^* \in \Delta : (y - x^*)^T F(x^*) \geq 0, \quad \forall y \in \Delta\}$$
- Many algorithms but not incentive-oriented: projection method[2], Tikhonov regularization[2], proximal point method [2], control method[3]

[2] Facchinei and Pang 2003.

[3] Allibhoy and Cortés 2023.

Extend Best Response Dynamics to general set K

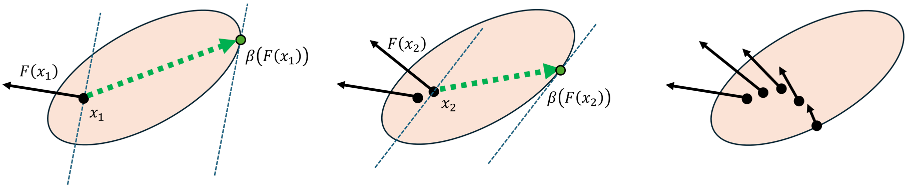
Definition (Best Response)

Let $K \subseteq \mathbb{R}^n$ be a compact and convex set. Given a vector $\pi \in \mathbb{R}^n$, the best response mapping $\beta(\pi)$ is defined as $\beta(\pi) = \arg\min_{y \in K} \; y^T \pi$.

Definition (Best Response Dynamics, BRD)

Given a population game F, the BRD is the differential inclusion,

$$\dot{x}(t) \in \beta(\pi(t)) - x(t), \quad \pi(t) = F(x(t)), \quad \forall t \geq 0.$$



Best Response Dynamics for solving Variational Inequalities

Definition (Strong Monotonicity / Monotonicity)

A function $F : K \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is strongly monotone (or monotone) on K if $\exists\ c > 0$ (or $c = 0$) s.t. $(x - y)^T (F(x) - F(y)) \geq c||x - y||^2, \quad \forall x, y \in K.$

Theorem (Convergence)

Consider the VI where $K \subseteq \mathbb{R}^n$ is compact and convex and $F : K \to \mathbb{R}^n$ is C^1 monotone. Then SOL(K, F) is globally asymptotically stable under the BRD: $\dot{x}(t) \in \beta(\pi(t)) - x(t)$ with $\pi(t) = F(x(t))$.

Proof.

- Existence of solutions
- Lyapunov function: $V = x^T F - \min_{y \in K} y^T F$
- Clarke's generalized gradient and envelope theorem

Three types of disturbances

Perfect cost function and perfect dynamics:

$$\dot{x}(t) \in \beta(\pi(t)) - x(t), \quad \pi(t) = F(x(t))$$

- cost disturbance: e.g., travel time increases due to accidents

$$\dot{x}(t) \in \beta(\tilde{\pi}(t)) - x(t), \quad \tilde{\pi}(t) = F(x(t)) + \xi(t)$$

- dynamics disturbance: e.g., following traffic police instructions

$$\dot{x}(t) \in \beta(\pi(t)) - x(t) + \varepsilon(t), \quad \pi(t) = F(x(t))$$

- state-dependent cost disturbance

$$\dot{x}(t) \in \beta(\tilde{\pi}(t)) - x(t), \quad \tilde{\pi}(t) = F(x(t)) + \delta(x(t))$$

Robustness Theorem I

Theorem (Dynamics disturbance and Cost disturbance)

Consider the VI with K compactly convex and F strongly monotone. If the BRD is subject to an admissible dynamics disturbance $\varepsilon(t)$ and a bounded cost disturbance $\xi(t)$ with a bounded derivative,

$$\dot{x}(t) \in \beta(\tilde{\pi}(t)) - x(t) + \varepsilon(t), \quad \tilde{\pi}(t) = F(x(t)) + \xi(t),$$

then

$$||x(t) - x^*|| \leq \omega(||x(0) - x^*||, t) + \gamma_1 \left(\max\left\{||\varepsilon||_\infty, ||\xi||_\infty\right\}\right) + \gamma_2(||\dot{\xi}||_\infty)$$

where x^* is the unique solution of VI(K, F), $\omega \in \mathcal{KL}$, and $\gamma_1, \gamma_2 \in \mathcal{K}$.

- Bounded disturbance $\rightarrow$ bounded perturbed distance.
  In particular, if $\varepsilon(t) \rightarrow 0$, $\xi(t) \rightarrow 0$, and $\dot{\xi}(t) \rightarrow 0$, then $x(t) \rightarrow x^*$

Robustness Theorem II

Theorem (State-dependent cost disturbance)

Consider the VI with K compactly convex and F strongly monotone. If the BRD is subject to a state-dependent cost disturbance $\delta(x)$,

$$\dot{x}(t) \in \beta(\tilde{\pi}(t)) - x(t), \quad \tilde{\pi}(t) = (F + \delta)(x(t)),$$

then, the dynamics converge to a new perturbed equilibrium point $\tilde{x}^*$ and

$$||x^* - \tilde{x}^*|| \leq \alpha_1^{-1}\left(h(\tilde{x}^*)\right),$$

where x^* is the unique solution of unperturbed VI(K, F) and $\alpha_1 \in \mathcal{K}_\infty$.

- $h(x) = \max\left\{(z - x)^T \delta(x) : z = \arg\min_{y \in K} \; y^T F(x)\right\}$
- δ is such that $F + \delta$ is strongly monotone

Robustness Theorem II: connection to perturbed BRD

- The perturbed best response dynamics is defined as

$$\dot{x}(t) \in \tilde{B}(F(x(t))) - x(t), \tilde{B}(F(x(t))) = \arg\min_{y\in\Delta} y^T F(x(t)) + H(y),$$

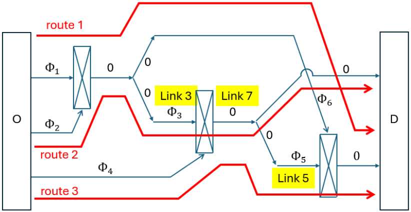
where H is strictly convex, usually selected as the entropy function.

- If solving optimization problem $\tilde{B}$ by Frank-Wolfe method, then

$$\begin{aligned}\dot{x}(t) &\in \arg\min_{y\in\Delta}\ y^T \left(F(x(t)) + \nabla H(x(t))\right) - x(t) \\ &= \beta((F + \nabla H)(x(t))) - x(t)\end{aligned}$$

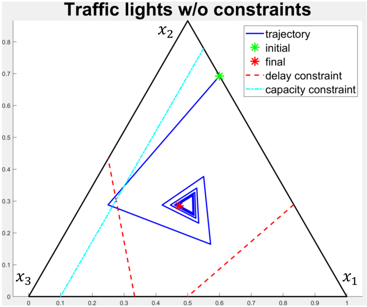
- We reproduce the convergence results for perturbed BRD. In addition, we derive a bound on the perturbed distance.

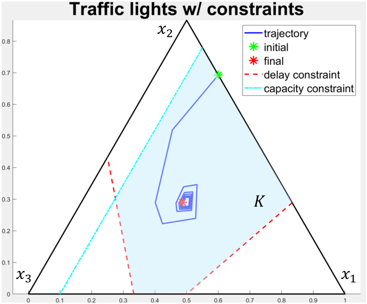
Traffic light network with constraints



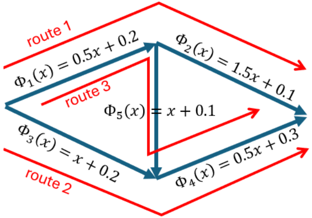
$$F(x) = \begin{bmatrix} 2x_1 + 3x_2 + x_3 \\ x_1 + 2x_2 + 3x_3 \\ 3x_1 + x_2 + 2x_3 \end{bmatrix}$$

$$K = \{x : x_2 + x_3 \leq 0.9, x_2 + 3x_3 \leq 2, x_3 + 3x_1 \leq 2, x \in \Delta\}$$



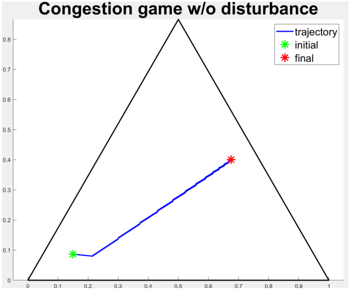


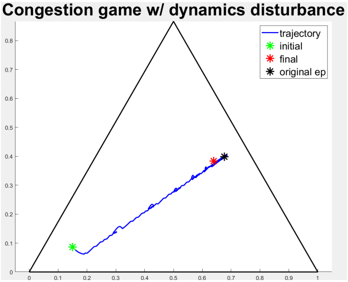
Robustness: dynamics disturbance



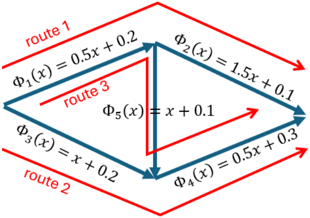
$$F(x) = \begin{bmatrix} 2x_1 + 0.5x_3 + 0.3 \\ 1.5x_2 + 0.5x_3 + 0.5 \\ 0.5x_1 + 0.5x_2 + 2x_3 + 0.6 \end{bmatrix}$$

$$\varepsilon(t) = [0.7\sin(0.1t), 0.7\cos(0.2t - 10), -0.7\sin(0.1t) - 0.7\cos(0.2t - 10)]^T$$





Robustness: cost disturbance



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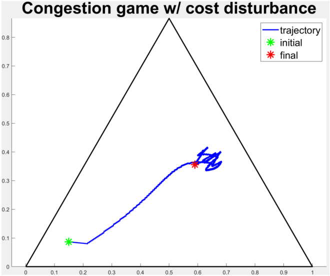
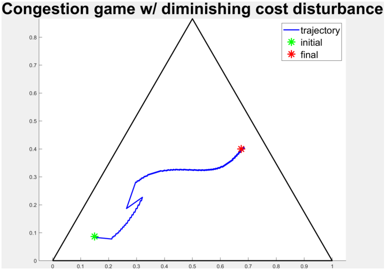
$$\xi_1(t) = [0.1\sin(0.01t), -0.05\cos(0.02t + 10), 0.15\sin(0.05t - 20)]^T$$

$$\xi_2(t) = 20e^{-0.01t}\xi_1(t)$$





Conclusions

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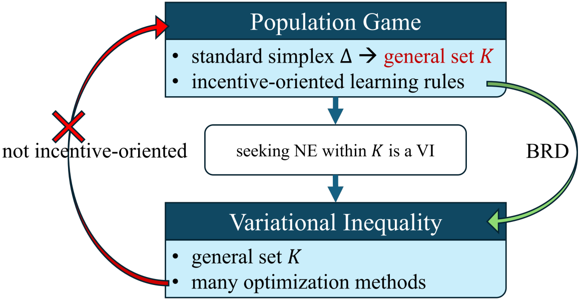

- Robustness analyses: bounds for perturbed distances
  - cost disturbance
  - dynamics disturbance
  - state-dependent cost disturbance

    $\Rightarrow$ draw a connection to perturbed BRD

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